December 9 , 2004

Department of Mathematics and Statistics
University of Saskatchewan

Math 238 Final Examination

Closed book. Calculators are allowed. You are not permitted to consult with your fellow students in any way. Time: 3 hours .

Part I

Question 1. (Three questions, each worth 1 pt) Determine whether the series converges or diverges. Find the sum of each convergent series. Be careful about the lower limits of summation.

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$$(a) \sum_{n=1}^{\infty} \sqrt{\frac{n^2-1}{n^2+1}},$$

$$(b) \sum_{n=0}^{\infty} \frac{7^{n+3}}{3^{2n-2}},$$

$$(c) \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{2^n}.$$

Question 2. (2 pts) Find the radius of convergence of each of the power series below:

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$$(a) \sum_{n=1}^{\infty} \frac{1}{n} x^{2n},$$

$$(c) \sum_{n=0}^{\infty} \frac{1}{2^n} (x-3)^{2n}$$

1 *Question 3.* (2 pts) Find the Maclaurin series for $f(x) = x^3 e^{x^2}$ and compute $f^{(1000)}(0)$, the 1000-th derivative of f at $x = 0$.

4 *Question 4.* (4 pts) Find the Taylor series of $f(x) = \cos x$ about $x = \frac{\pi}{2}$

2 *Question 5.* (2 pts) Solve the initial value problem $yy' = \sin^2 t, \quad y(0) = \sqrt{3}$
Hint: $\sin^2 t = \frac{1-\cos 2t}{2}$

Question 6. (2 pts) Solve the initial value problem $y'' + y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = 1$

Question 7. (3 pts) Find a general solution of $y'' - 2y' + y = e^t \sin t$.

Question 8. (2 pts) Suppose we know two solutions y_1 and y_2 of the equation $y'' + p(t)y' + q(t)y = 0$, where $p(t)$ and $q(t)$ are continuous functions on the interval $[a, b]$. Suppose in addition that $y_1(t_0) = y_2(t_0) = 0$ at $a < t_0 < b$. Show that $\{y_1, y_2\}$ do not form a fundamental set. Give a precise argument. If you are using a theorem, state the theorem in its entirety, with all the necessary assumptions.

Part II

Question 9. (4 pts) A cylindrical buoy 60 cm in diameter stands in water with its axis vertical. When depressed slightly and released, it is found that the period of vibration is 2 sec. Determine the weight of the buoy. Hint: By Archimedes's

principle, the buoyancy force equals the weight of the water displaced by the body. Some, possibly useful, constants: $g = 9.81m/sec^2$, $\rho_{water} = 1000kg/m^3$.

Question 10. (3 pts) Show that $xy' + y + 4 = 0$ is exact and then solve it.

Question 11. (3 pts) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Hint: Use the partial fraction decomposition.

Part III

Question 12. (3 pts) How many terms in the Maclaurin series for $f(x) = e^{-x^2}$ guarantee a truncation error of less than 10^{-5} for all x in the interval $0 \le x \le 2$?

Question 13. (4 pts) Show that the sequence $c_1 = 2, c_{n+1} = \frac{1}{3-c_n}$, $n \ge 1$ is convergent and find its limit.
Hint: Show by induction that $0 < c_n \le 2$, for all $n \ge 1$.

Question 14. (3 pts) Find a general solution of the equation:

$$y'' + 2\gamma y' + \omega_0^2 y = \frac{f_0}{m} e^{\alpha t} \cos \beta t$$

by first solving the same equation with the right hand side replaced by $\frac{f_0}{m} e^{\alpha t} e^{i\beta t}$ and then taking the real part of the solution. Assume $\boxed{\omega_0^2 > \gamma^2}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{(-4)^n}{n!}$$

$$2^3 2^4 = 2^{\wedge}$$

$$(5^7)(8^7) = (40)^7 ?$$

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8.00
32.74
65.91